Example: The Surface Integral

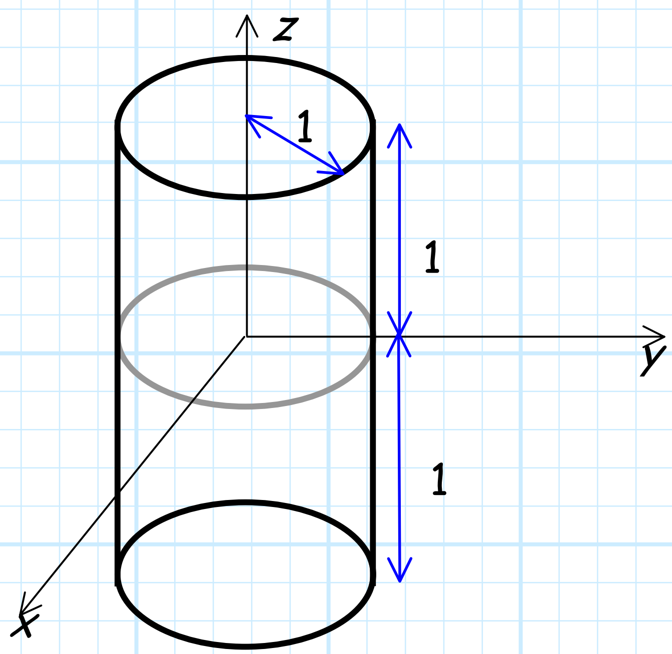
Consider the vector field:

$$\mathbf{A}(\bar{r}) = x \ \hat{a}_x$$

Say we wish to **evaluate** the surface integral:

$$\iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds}$$

where S is a **cylinder** whose axis is aligned with the z-axis and is centered at the origin. This cylinder has a **radius** of 1 unit, and extends 1 unit below the x-y plane and one unit above the x-y plane. In other words, the cylinder has a **height** of 2 units.



This is a **complex**, **closed** surface. We will define the **top** of the cylinder as surface S_1, the **side** as S_2, and the **bottom** as S_3. The surface integral will therefore be evaluated as:

$$\iint_S \mathbf{A}(\bar{r}_s)\cdot\overline{ds} = \iint_{S_1} \mathbf{A}(\bar{r}_s)\cdot\overline{ds_1} + \iint_{S_2} \mathbf{A}(\bar{r}_s)\cdot\overline{ds_2} + \iint_{S_3} \mathbf{A}(\bar{r}_s)\cdot\overline{ds_3}$$

Step 1: Determine $\overline{ds}$ for the surface S.

Let's define $\overline{ds}$ as pointing in the direction outward from the closed surface.

S_1 is a **flat plane** parallel to the *x-y* plane, defined as:

$$0 \le \rho \le 1 \quad 0 \le \phi \le 2\pi \quad z = 1$$

and whose outward pointing $\overline{ds}$ is:

$$\overline{ds_1} = \overline{ds_z} = \hat{a}_z\ \rho\, d\rho\, d\phi$$

S_2 is a **circular cylinder** centered on the *z*- axis, defined as:

$$\rho = 1 \quad 0 \le \phi \le 2\pi \quad -1 \le z \le 1$$

and whose outward pointing $\overline{ds}$ is:

$$\overline{ds_2} = \overline{ds_\rho} = \hat{a}_\rho\ \rho\, dz\, d\phi$$

S_3 is a **flat plane** parallel to the *x-y* plane, defined as:

$$0 \leq \rho \leq 1 \quad 0 \leq \phi \leq 2\pi \quad z = -1$$

and whose outward pointing $\overline{ds}$ is:

$$\overline{ds_3} = -\overline{ds_z} = -\hat{a}_z \; p \, dp \, d\phi$$

Step 2: Evaluate the dot product $\mathbf{A}(\bar{r}_s) \cdot \overline{ds}$.

$$\begin{aligned} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_1} &= x \; \hat{a}_x \cdot \hat{a}_z \; \rho \, d\rho \, d\phi \\ &= x \, (0) \, \rho \, d\rho \, d\phi \\ &= 0 \end{aligned}$$

$$\begin{aligned} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} &= x \; \hat{a}_x \cdot \hat{a}_p \; \rho \, dz \, d\phi \\ &= x \left(\cos\phi\right) \rho \, dz \, d\phi \end{aligned}$$

$$\begin{aligned} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_3} &= -x \; \hat{a}_x \cdot \hat{a}_z \; \rho \, d\rho \, d\phi \\ &= -x \, (0) \, \rho \, d\rho \, d\phi \\ &= 0 \end{aligned}$$

Look! Vector field $A(\bar{r})$ is **tangential** to surface S_1 and S_3 for all points on surface S_1 and S_3! Therefore:

$$\begin{aligned} \iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds} &= \iint_{S_1} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_1} + \iint_{S_2} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} + \iint_{S_3} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_3} \\ &= 0 + \iint_{S_2} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} + 0 \\ &= \iint_{S_2} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} \end{aligned}$$

Step 3: Write the resulting scalar field using the same coordinate system as $\overline{ds}$.

The differential vector $\overline{ds_\rho}$ is expressed in **cylindrical** coordinates, therefore we must write the **scalar** integrand using cylindrical coordinates.

We know that:

$$x = \rho \cos\phi$$

Therefore:

$$\begin{aligned} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} &= x(\cos\phi)\,\rho\, dz\, d\phi \\ &= \rho\cos\phi(\cos\phi)\,\rho\, dz\, d\phi \\ &= \rho^2\cos^2\phi\ dz\, d\phi \end{aligned}$$

Step 4: Evaluate the scalar field using the coordinate **equality** that described surface S.

Every point on S_2 has the coordinate value $\rho = 1$. Therefore:

$$\begin{aligned} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} &= \rho^2\cos^2\phi\ dz\, d\phi \\ &= 1^2\cos^2\phi\ dz\, d\phi \\ &= \cos^2\phi\ dz\, d\phi \end{aligned}$$

Step 5: Determine the **limits of integration** from the **inequalities** that describe surface S.

For S_2 we know that $0 \le \phi \le 2\pi \quad -1 \le z \le 1$.

Therefore:

$$\iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds} = \iint_{S_2} \mathbf{A}(\bar{r}_s) \cdot \overline{ds_2} = \int_0^{2\pi} \int_{-1}^{1} \cos^2\phi \; dz \, d\phi$$

Step 6: Integrate the remaining function of **two** coordinate variables.

Using **all** the results determined above, the surface integral becomes:

$$\begin{aligned}
\iint_S \mathbf{A}(\bar{r}_s) \cdot \overline{ds} &= \int_0^{2\pi} \int_{-1}^{1} \cos^2\phi \; dz \, d\phi \\
&= \int_0^{2\pi} \cos^2\phi \; d\phi \int_{-1}^{1} dz \\
&= (\pi - 0)(1 - (-1)) \\
&= \mathbf{2\pi}
\end{aligned}$$